BRIEF NOTES ON AG I: BASIC LANGUAGE

SHIZHANG LI

Slogan:"language of scheme = globalization of commutative algebra".

1. (Pre-)sheaves and (Locally) ringed spaces

Definition 1.1. topological space = a set X + a class of subsets $\{U \subset X\}$ that we call "open subsets" which satisfies:

- the subsets $\emptyset$ and X belong to this class;
- this class of subsets is stable under finite intersection and arbitrary union.

Exercise 1.2. Find the definition of "category" and "functor".

Definition 1.3. Let X be a topological space, then a presheaf of sets/abelian groups/rings/...on X is a functor $\mathcal{F}\colon \{\text{opens}\}^{op} \to$ Sets/Abelian Groups/Rings. Concretely this means a rule $\mathcal{F}$ which assigns a set/ab gp/ring to an open subset $U \mapsto \mathcal{F}(U)$ called "the sections of $\mathcal{F}$ on U", and "restriction" maps for each $U \subset V$: $res^V_U\colon \mathcal{F}(V) \to \mathcal{F}(U)$ such that

- for each $U \subset V \subset W$ we have $res^V_U \circ res^W_V = res^W_U$; and
- $res^U_U = id$.

A map of presheaves $f\colon \mathcal{F} \to \mathcal{G}$ is what you think it is: a bunch of $f(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ commuting with the restriction maps (or natural transformations if you think of presheaves as functors).

We also use $\Gamma(U, \mathcal{F})$ to denote $\mathcal{F}(U)$.

A sheaf of blabla is a presheaf of blabla satisfying the following condition: $\forall$ open cover $U = \cup_{i \in I} U_i$, the natural map $\mathcal{F}(U) \to Eq\left(\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)\right)$ is an isomorphism. The two arrows will send a tuple $(s_i)_{i \in I}$ to two tuples whose values on (i,j) are given by s_i and s_j respectively. Maps of sheaves = view them as presheaves and consider maps of such.

Remark 1.4. (1) Concretely the sheaf condition says that a section $s \in \mathcal{F}(U)$ is uniquely given by a tuple $(s_i \in \mathcal{F}(U_i))_{i \in I}$ satisfying the condition that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, people often say "these s_i's glue to s".

(2) Notice that the empty set is covered by empty family ($I = \emptyset$), and empty product (if exists) is always computed by final object, so the above implicitly says that $\mathcal{F}(\emptyset)$ = final object in the relevant category.

Definition 1.5. The stalk of a (pre-)sheaf $\mathcal{F}$ at a point $x \in X$ is defined as the following filtered colimit $\mathcal{F}_x := colim_{x \in U} \mathcal{F}(U)$ with transition maps given by restriction.

Exercise 1.6. Convince yourself that $\forall x \in U \subset X$, $\exists \mathcal{F}(U) \to \mathcal{F}_x$. And if $\mathcal{F}$ is a sheaf, then $\forall U \subset X$ the natural map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective.

Definition 1.7. A map of presheaves $\mathcal{F} \to \mathcal{G}$ is injective/surjective if the induced maps $\mathcal{F}(U) \to \mathcal{G}(U)$ are injective/surjective for all opens U.

Exercise 1.8. (1) Find the definition of "abelian category", and prove that {presheaves of abelian groups on a topological space X} is an abelian category with the above notion of inj/surj.

(2) If $\mathcal{F} \to \mathcal{G}$ is an inj/surj map of pre-sheaves, then for any $x \in X$ the induced map between stalks $\mathcal{F}_x \to \mathcal{G}_x$ is inj/surj.

(3) Find an example of maps between pre-sheaves which is neither inj nor surj, but whose induced map between stalks are all isomorphisms.

Definition 1.9. A map of sheaves $\mathcal{F} \to \mathcal{G}$ is injective/surjective if the induced map between stalks $\mathcal{F}_x \to \mathcal{G}_x$ are injective/surjective for all $x \in X$.

Exercise 1.10. (1) Show that a map of sheaves is an isomorphism iff it's both inj and surj.
(2) Show that the kernel presheaf of a map of sheaves is already a sheaf.
(3) Give an example of a map of sheaves which is surjective as sheaves but not surjective as presheaves.

Construction 1.11 (Sheafification)**.** Let $\mathcal{F}$ be a presheaf on X, define another presheaf $\mathcal{F}^+$ as follows: for any $U \subset X$ open, we let $\mathcal{F}^+(U)$ be the set/ab gp/ring of maps

$$s\colon U \to \coprod_{x \in U} \mathcal{F}_x,$$

with $s(x) \in \mathcal{F}_x$ and such that $\forall u \in U$, $\exists$ open neighborhood $u \in U_u \subset U$ and an element $t \in \mathcal{F}(U_u)$ satisfying the following equality

$$s(u') = t_{u'},\ \forall u' \in U_u.$$

Exercise 1.12. (1) Show that one may alternatively define $\mathcal{F}^+(U)$ as the equivalence classes of pairs $(U = \cup_{i \in I} U_i, s_i \in \mathcal{F}(U_i))$, where $(U = \cup_{i \in I} U_i, s_i \in \mathcal{F}(U_i))$ and $(U = \cup_{j \in J} U'_j, s'_j \in \mathcal{F}(U'_j))$ are equivalent iff $\forall (i,j)$, $\exists$ an open cover $U_i \cap U'_j = \cup_{k \in K} U^{(k)}_{i,j}$ and equalities $s_i|_{U^{(k)}_{i,j}} = s'_j|_{U^{(k)}_{i,j}}$.
(2) Show that $\mathcal{F}^+$ is a sheaf.
(3) Show that there is a natural map $\iota\colon \mathcal{F} \to \mathcal{F}^+$ inducing isomorphisms on stalks.
(4) Show that $\forall$ sheaf $\mathcal{G}$ and $\forall$ presheaf map $f\colon \mathcal{F} \to \mathcal{G}$, $\exists!$ map of sheaves $\widetilde{f}\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\widetilde{f} \circ \iota = f$.
(5) Show that {sheaves of abelian groups on a topological space X} is an abelian category with cokernel given by sheafification of the presheaf cokernel (and kernel given by presheaf kernel which is already a sheaf by an exercise above).
(6) Let $f\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves of abelian groups, give an explicit description of $coker(f)$.

Definition 1.13. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A map of ringed spaces $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f\colon X \to Y$ and a bunch of ring maps $f^\sharp(V \subset Y)\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ commuting with restriction maps (exercise: please make this precise).

Exercise 1.14. Let $f\colon X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. Then define a presheaf $f_*\mathcal{F}$ on Y by $f_*\mathcal{F}(V \subset Y) := \mathcal{F}(f^{-1}V)$. Show that $f_*\mathcal{F}$ is a sheaf if $\mathcal{F}$ is so. So in the definition above, we may interpret the second part of data as a map of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Definition 1.15. A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that $\forall x \in X$ the stalks $\mathcal{O}_{X,x}$ are local rings. A map of locally ringed spaces $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of ringed spaces with the following additional requirement: $\forall x \in X$, the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a map of local rings.

Definition 1.16. There is a natural notion of restriction of a (pre)sheaf to an open subset, and we define an open (locally) ringed subspace as an open topological space equipped with the restriction of the sheaf of rings.

2. Schemes and (Quasi-)coherent sheaves

The word "ring" shall always mean commutative unital ring.

Definition 2.1. Let A be a ring, the set of prime ideals in A is denoted by $Spec(A)$. Given an ideal $I \subset A$, define a subset $V(I) \subset Spec(A)$ by $V(I) :=$ {prime ideals which contain I}.

The unit ideal $(1) = A \subset A$ is *NOT* a prime ideal, so $V(A) = \emptyset$. An element (or a point) in $Spec(A)$ is a prime ideal $\mathfrak{p} \subset A$. It's straightforward to check that there is a bijection between sets $Spec(A/I) \simeq V(I)$. Below we shall define the so-called "Zariski topology" on it and equip the topological space with a locally ringed space structure.

Definition 2.2 (Zariski topology)**.** We define the Zariski topology on $Spec(A)$ by declaring closed subsets are given by $V(I)$, where $I \subset A$ is an ideal. One particular class of opens in $Spec(A)$, also called "standard opens", are given by: Let $f \in A$, then denote $D(f) = Spec(A) \setminus V(f)$[1]. Equivalently, we have $D(f) := \{\mathfrak{p} \text{ such that } f \notin \mathfrak{p}\} \subset Spec(A)$.

[1] $V(f)$ shall be a short-hand for $V((f))$.

Exercise 2.3. (1) Check that this defines a topology.

(2) Check that standard opens $\{D(f); f \in A\}$ form a basis of the topology: concretely this says that any open neighborhood U of a point $\mathfrak{p} \in Spec(A)$ contains a standard open neighborhood $\mathfrak{p} \in D(f) \subset U$.

(3) Show that a prime ideal $\mathfrak{p}$ gives rise to a closed point in $Spec(A)$ iff it's a maximal ideal.

(4) Check that the set bijection $V(I) \simeq Spec(A/I)$ is a homeomorphism when we equip the source by its subspace topology inherited from $V(I) \subset Spec(A)$.

(5) Let $f \in A$. Show that there is a natural homeomorphism $D(f) \simeq Spec(A[1/f])$.

(6) Let $f, g \in A$. Show that $D(g) \subset D(f)$ iff $V(f) \subset V(g)$ iff $g^e \in (f)$ for some e iff f is invertible in $A[1/g]$.

(7) Show that $D(f) \subset \bigcup_{i \in I} D(g_i)$ iff $\{g_i; i \in I\}$ generates unit ideal in $A[1/f]$.

(8) Check that this topology makes $Spec(A)$ quasi-compact: any open cover of $Spec(A)$ can be refined by a finite open cover.

Definition 2.4 (Structure sheaf)**.** Let A be a ring and let $X = Spec(A)$. Define a presheaf $\mathcal{O}_X$ by declaring its sections on an open $U \subset Spec(A)$ to be: The ring $\mathcal{O}_X(U)$ of maps $f\colon U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ with $f(\mathfrak{p}) \in A_{\mathfrak{p}}$ and such that $\forall \mathfrak{p} \in U$, $\exists$ open neighborhood $\mathfrak{p} \in D(f) \subset U$ and an element $t \in A[1/f]$ satisfying the following equality

$$f(\mathfrak{p}') = t \in A_{\mathfrak{p}'},\ \forall \mathfrak{p}' \in D(f).$$

Definition 2.5 (Quasi-coherent sheaf)**.** Keep notation as above, and let M be an A-module. Define a presheaf $\widetilde{M}$ on X by declaring its sections on an open $U \subset Spec(A)$ to be: The $\mathcal{O}_X(U)$-module of maps $f\colon U \to \bigsqcup_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ with $f(\mathfrak{p}) \in M_{\mathfrak{p}}$ and such that $\forall \mathfrak{p} \in U$, $\exists$ open neighborhood $\mathfrak{p} \in D(f) \subset U$ and an element $t \in M[1/f]$ satisfying the following equality

$$f(\mathfrak{p}') = t \in M_{\mathfrak{p}'},\ \forall \mathfrak{p}' \in D(f).$$

Exercise 2.6. (0) Let X be a topological space and $\mathcal{O}_X$ be a presheaf of rings on X. Formulate the definition of "the category of presheaves of $\mathcal{O}_X$-modules", then show that $\widetilde{M}$ defined above is an example of such. (Crucially, one needs to consider the compatibility between restriction maps on $\mathcal{O}_X$ and $\mathcal{F}$.)

Now let notation be as above, so that A is a ring with $X = Spec(A)$.

(1) Show that $\mathcal{O}_X$ and $\widetilde{M}$ defined above are sheaves.

(2) Show that $\mathcal{O}_X|_{D(f)} = \mathcal{O}_{Spec(A[1/f])}$ and $\widetilde{M}|_{D(f)} = \widetilde{M[1/f]}$; here we have used the homeomorphism $D(f) \simeq Spec(A[1/f])$.

(3) Show that $\mathcal{O}_X(D(f)) = A[1/f]$ and $\widetilde{M}(D(f)) = M[1/f]$.

(4) Show that $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$ and $(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$.

(5) Show that a map of A-modules $M \to N$ is inj/surj iff the induced map of sheaves $\widetilde{M} \to \widetilde{N}$ is so.

Definition 2.7 (Spectrum of a ring)**.** Let A be a ring, then the pair $(Spec(A), \mathcal{O}_X)$ defines a locally ringed space, called "the spectrum of A". From now on we will slightly abuse notation and use $Spec(A)$ to denote this locally ringed space. The collection of $Spec(A)$'s defines a full subcategory of $\{$locally ringed space$\}$, objects of which we call *affine schemes* (so an affine scheme is the same as a locally ringed space isomorphic to $Spec(A)$, for a unique choice of ring A by the above exercise).

Those sheaves of $\mathcal{O}_X$-modules of the shape $\widetilde{M}$ are called "*quasi-coherent sheaves*".

Suppose A is Noetherian. Then we call $\widetilde{M}$ a *coherent sheaf* if M is finitely generated.

Exercise 2.8. (1) Let A be a ring with $X = Spec(A)$. Show that the assignment $M \mapsto \widetilde{M}$ gives rise to an *exact* equivalence

$$\{A\text{-mod's}\} \cong \{\text{Quasi-coherent sheaves on } Spec(A)\}.$$

Concretely: Given two A-modules M and N, the two homomorphism groups $Hom_A(M, N)$ and $Hom_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ are isomorphic via $\widetilde{(-)}$ and $\Gamma(X, -)$. And inj/surj on each side is reflected on the other side: a map $M \to N$ is inj/surj iff $\widetilde{M} \to \widetilde{N}$ is inj/surj.

(2) Show that the assignment $A \mapsto Spec(A)$ gives rise to an equivalence:

$$\{\text{Rings}\}^{op} \cong \{\text{Affine schemes}\}.$$

The inverse is given by taking global section of structure sheaf.

Definition 2.9. A locally ringed space is called a scheme if it admits an open cover by affine schemes. Maps between two schemes is defined as that of them viewed as LRS's. So {schemes} is a full subcategory of {locally ringed spaces}.

Exercise 2.10. Let X be a scheme, and let $Y = Spec(A)$ be an affine scheme. Show that there is a bijection between sets $Map_{Schemes}(X,Y) \cong Hom_{Rings}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$.

Exercise 2.11. (1) Let X be an affine scheme, and let U be an open locally ringed subspace of X, show that U is a scheme.

(2) Let X be a scheme, and let U be an open locally ringed subspace of X, show that U is a scheme.

Definition 2.12. These U's are called open subschemes of X.

(3) In the situation of (2) and denote the inclusion map between schemes by $(j, j^\sharp): U \to X$. Let $x \in U \subset X$, check that the natural map between stalks $j^\sharp|_x : \mathcal{O}_{X,x} \to \mathcal{O}_{U,x}$ is an isomorphism.

In particular, let $A \to B$ be a ring homomorphism giving rise to an open subscheme $Y = Spec(B) \subset X = Spec(A)$, and let $\mathfrak{q} \in Y$ with its image $\mathfrak{p} \in X$. Then the natural map $A \to B$ induces an isomorphism between localizations $A_\mathfrak{p} \xrightarrow{\cong} B_\mathfrak{q}$. In particular such a ring homomorphism $A \to B$ is flat.

(3.5) In the above setup, the localization process $B \to B_\mathfrak{q}$ is the same as joining elements $f \in A \setminus \mathfrak{p}$ (since an open neighborhood of $\mathfrak{q}$ in Y contains an open neighborhood of $\mathfrak{p}$ in X.) Therefore we also have $A_\mathfrak{p} \otimes_A B \xrightarrow{\cong} B_\mathfrak{q}$. Using this, one can show $B \xrightarrow{\cong} B \otimes_A B$ via either $id \otimes 1$ or $1 \otimes id$.

(4) Let $A \to B$ be a ring homomorphism giving rise to an open subscheme $Y = Spec(B) \subset X = Spec(A)$. Let M be an A-module corresponding to the quasi-coherent sheaf $\widetilde{M}$ on X. Then the restriction $\widetilde{M}_Y$ is a quasi-coherent sheaf corresponding to the B-module $M \otimes_A B$.

Definition 2.13 (Quasi-coherent sheaf, again)**.** Let X be a scheme. A sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ is called a quasi-coherent sheaf if there is an open cover of $X = \cup U_i = Spec(A_i)$ by affine open subschemes such that $\forall i$, $\mathcal{F}|_{U_i}$ is quasi-coherent.

Exercise 2.14. (1) Show that this definition agrees with the previous one when X is an affine scheme. Namely if X is an affine scheme with an open cover by affine opens $U_i = Spec(A_i)$. Then a sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ is of the form $\widetilde{M}$ for some A-module M iff its restriction to each U_i is of the form $\widetilde{M_i}$ for some A_i-module M_i.

(2) Let $f : X \to Y$ be a map of schemes, and let $\mathcal{F}$ be a quasi-coherent sheaf. Define a quasi-coherent sheaf $\mathcal{G}$ on X with stalks at points $x \in X$ given by $\mathcal{F}_x \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$, (Hint: define the sheaf of maps from opens to disjoint union of these stalks, blabla...) so that on affine open $U \subset X$ which maps into an affine open $V \subset Y$ by f, the $\mathcal{G}|_U$ is associated with the module $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$.

Definition 2.15. The $\mathcal{G}$ produced this way is denoted by $f^*(\mathcal{F})$.

Definition 2.16 (Quasi-coherent sheaf of ideals)**.** Let X be a scheme. A *quasi-coherent sheaf of ideals* is a quasi-coherent $\mathcal{O}_X$-subsheaf of $\mathcal{O}_X$.

Exercise 2.17. Please check the following.

(1) When $X = Spec(A)$ is an affine scheme, then a quasi-coherent sheaf of ideals on X is the same as an ideal in A.

(2) Let $X = Spec(A)$ be an affine scheme, let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X corresponding to an ideal $I \subset A$. Let $Y = Spec(B)$ be an affine open subscheme of X, and $Y \to X$ corresponds to a ring homomorphism $f : A \to B$. Then the quasi-coherent sheaf of ideals $\mathcal{I}|_Y$ on Y corresponds to the ideal $f(I)B \subset B$.

(3) Let X be a scheme, let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. Show that the subset $Z \subset X$ defined by $\{x \in X | \mathcal{I}_x \to \mathcal{O}_{X,x}$ is *NOT* a bijection$\}$ is closed.

(4) Keep notation as (3), show that the sheaf of rings $\mathcal{O}_X/\mathcal{I}$ is supported on Z (namely restricts to the sheaf of zero ring away from Z) and can be described by: Sections on an open $U \cap Z$ is given by set map

$$s : U \cap Z \to \bigsqcup_{x \in U \cap Z} \mathcal{O}_{X,x}/\mathcal{I}_x$$

with $s(x) \in \mathcal{O}_{X,x}/\mathcal{I}_x$ and the similar condition as before (OK, formulating this is part of the exercise).

(5) Keep notation as (4), finally show that $(Z, \mathcal{O}_X/\mathcal{I})$ is a scheme.

(Hint: It suffices to prove (3)-(5) when $X = Spec(A)$ is affine.)

Definition 2.18 (Closed subschemes)**.** Let X be a scheme, then $(Z, \mathcal{O}_X/\mathcal{I})$'s arising from quasi-coherent sheaves of ideals in the above manner are called *closed subschemes of* X.

In other words, closed subschemes of X are the same as closed locally ringed spaces corresponding to quasi-coherent sheaves of ideals.

A map of schemes $i\colon Z \to X$ is called *closed immersion* if it induces an isomorphism between Z and a closed subscheme of X. Alternatively one may define closed immersion by requiring i be a closed immersion on the topological spaces and the map of sheaves $i^\sharp$ is surjective. (Leave as an exercise to check this. Hint: the corresponding quasi-coherent sheaf of ideals is simply the kernel of $i^\sharp$.)

3. General discussion

Question 3.1. What are some examples of schemes except the affine ones?

Exercise 3.2. Let k be a field, and let $X = Spec(k[x,y])$. The maximal ideal (x,y) defines a closed point $x \in X$. Consider the open subscheme $U \coloneqq X \setminus \{x\}$, show that U is *NOT* an affine scheme.

Definition 3.3. Open subschemes of an affine scheme are defined to be quasi-affine schemes.

Here's another sort of non-affine schemes.

Definition 3.4. A non-negatively graded ring is a ring S together with a decomposition $S = \bigoplus_{n\in\mathbb{N}} S_n$ with S_n's stable under addition and the multiplication on S induces maps $S_n \times S_m \to S_{n+m}$. In particular, S_0 is a subring.

Elements in S_n are called homogeneous elements of degree n, homogeneous elements are those for some n, homogeneous ideals I are ideals of S which satisfies $I = \bigoplus_{n\in\mathbb{N}} I \cap S_n$, homogeneous prime ideals are prime and homogeneous ideals. The ideal of positive elements is $S_+ \coloneqq \bigoplus_{n>0} S_n$.

Let $f \in S$ be a homogeneous element. Then $S[1/f]$ is a $\mathbb{Z}$-graded ring, we denote its degree 0 part by $S_{(f)}$.

Let $\mathfrak{p} \subset S$ be a homogeneous prime ideal, we have an associated $\mathbb{Z}$-graded localization $T^{-1}S$ where T is the multiplicative set of homogeneous elements in $S \setminus \mathfrak{p}$. Define $S_{(\mathfrak{p})}$ as the degree 0 part of $T^{-1}S$.

Construction 3.5 (Proj construction)**.** Let $S = \bigoplus_{n\in\mathbb{N}} S_n$ be a non-negatively graded ring. Define a ringed space $(Proj(S), \mathcal{O})$ by:

- The underlying set $Proj(S) \coloneqq \{$homogeneous prime ideals in S not containing $S_+\}$.
- The topology is defined by declaring the subsets of the form $V_+(I) \coloneqq \{\mathfrak{p}$ such that $I \subset \mathfrak{p}\}$ (where I is a homogeneous ideal) are closed.
- Define the structure sheaf $\mathcal{O}$ by: For any open $U \subset Proj(S)$, define $\mathcal{O}_U$ to be the set of maps $s\colon U \to \bigsqcup_{\mathfrak{p}\in U} S_{(\mathfrak{p})}$ with $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and such that $\forall \mathfrak{p} \in U$, $\exists$ open neighborhood $\mathfrak{p} \in V \subset U$ and homogeneous elements $a, f \in S$ of the same degree such that $\forall \mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{p})}$.

Exercise 3.6. (1) Obviously $Proj(S) \subset Spec(S)$ is a subset. Show that the topology above agrees with the subspace topology induced from the Zariski topology on $Spec(S)$. Also check that the $\mathcal{O}$ defined above is a sheaf of rings.

(2) For any homogeneous element $f \in S_+$, define $D_+(f) \coloneqq \{\mathfrak{p} | f \notin \mathfrak{p}\} \subset Proj(S)$. Show that these form a basis of $Proj(S)$.

(3) Show that the following graded ring maps $S \to S[1/f] \leftarrow S_{(f)}$ induces homeomorphism $D_+(f) \xleftarrow{\simeq} Spec(S[1/f]) \xrightarrow{\simeq} Spec(S_{(f)})$.

(4) Let $g \in S$ be a homogeneous element. Show that the graded ring map $S \to S/g$ induces a homeomorphism $V_+(g) \simeq Proj(S/g)$.

(5) Combining (3) and (4), show that if $f, g \in S_+$ with $D_+(g) \subset D_+(f)$, then f is a unit in $S[1/g]$ and $f^{|g|}/g^{|f|}$ is a unit in $S_{(g)}$, and this induces an isomorphism $S_{(f)}[(g^{|f|}/f^{|g|})^{-1}] \xrightarrow{\cong} S_{(g)}$.

The following exercise aims at proving that the ringed space $Proj(S)$ defined above is a scheme.

Exercise 3.7. (1) Prove the following lemma:

Lemma. *Let X be a topological space, and let $\mathcal{F}, \mathcal{G}$ be two sheaves of sets/ab gps/rings. Suppose there is a basis of opens $\mathfrak{B}$ and suppose $\forall U \in \mathfrak{B}$, there is a map $\mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps from U_1 to U_2 whenever $U_2 \subset U_1, U_i \in \mathfrak{B}$. Then these maps extend uniquely to a map of sheaves $\mathcal{F} \to \mathcal{G}$.*

(2) Let's apply the above lemma: The X there is our $D_+(f)$, the basis of opens $\mathfrak{B}$ are the opens $D_+(g) \subset D_+(f)$. Using the homeomorphism $D_+(f) \simeq Spec(S_{(f)})$, the structure sheaf of the latter defines a sheaf of rings $\mathcal{F}$. Using the inclusion $D_+(f) \subset Proj(S)$, the sheaf defined above restricts to a sheaf of rings $\mathcal{G}$. Check the following: For each $U \coloneqq D_+(g) \subset D_+(f)$, we have $\mathcal{F}(U) = S_{(g)}$ which has a natural map to $\mathcal{G}(U)$. Check that the induced map $\mathcal{F} \to \mathcal{G}$ induces isomorphisms on all stalks, hence we get an isomorphism of ringed spaces $(D_+(f), \mathcal{O}_{Proj(S)}|_{D_+(f)}) \xrightarrow{\cong} (Spec(S_{(f)}), \mathcal{O}_{Spec(S_{(f)})})$.

Definition 3.8. Combining these exercises we see that $Proj(S)$ is a scheme, called the *projective spectrum of S*.

Example 3.9. Let A be a ring, then the n-dimensional projective space $\mathbb{P}^n_A$ over A is defined to be the projective spectrum of the non-negatively graded ring $S \coloneqq A[X_0, \ldots, X_n]$ where all X_i have degree 1.

Exercise 3.10. Show that $\mathbb{P}^n_A$ is quasi-affine iff $n = 0$.

Construction 3.11 (Proj construction, this time for graded modules)**.** Let S be a non-negatively graded ring. A graded S-module M is an S-module together with a decomposition $M = \bigoplus_{n \in \mathbb{Z}} M_n$ where each M_n is stable under addition and the module action gives rise to maps $S_n \times M_m \to M_{n+m}$.

Keep notation as above, for each homogeneous prime ideal $\mathfrak{p} \in Proj(S)$, define $M_{(\mathfrak{p})}$ as the degree 0 part in the localization $T^{-1}M$ where T is what appears when defining $S_{(\mathfrak{p})}$. Similarly for any homogeneous element $f \in S$, define $M_{(f)}$ to be the degree 0 part in $M[1/f]$.

Define a sheaf $\widetilde{M}$ by the rule that its section on an open $U \subset Proj(S)$ is given by functions $U \to \bigsqcup_{\mathfrak{p} \in U} M_{(\mathfrak{p})}$, with ... such that ... It's not I'm feeling lazy or anything, but please complete the above sentence. In the end we get a sheaf of $\mathcal{O}_X$-modules, which is quasi-coherent, and its restriction to any $D_+(f)$ is given by $\widetilde{M_{(f)}}$.

Exercise 3.12. Let A be a ring, and let $S = A[X_0, \ldots, X_n]$ be the graded ring defining $\mathbb{P}^n_A$. Define a graded S-module $S(m)$ whose underlying module is S itself, but with a shift of grading: $S(m)_i = S_{m+i}$. Compute $\widetilde{S(m)}(Proj(S))$.

Next we discuss some aspects of the category of schemes.

Exercise 3.13. (0) Find the definition of fiber products in a category.

(1) Let A, B, C be rings, let $f\colon A \to B$ and $g\colon A \to C$ be ring homomorphisms. These maps correspond to maps of schemes $Spec(B) \to Spec(A)$ and $Spec(C) \to Spec(A)$. Show that the fiber product $Spec(B) \times_{Spec(A)} Spec(C)$ is given by $Spec(B \otimes_A C)$ together with natural maps to $Spec(B)$ corresponding to $B \xrightarrow{id \otimes 1} B \otimes_A C$ (and similarly for $Spec(C)$).

Definition 3.14. Given a scheme X with a point $x \in X$, then the residue field of the local ring $\mathcal{O}_{X,x}$ is denoted $\kappa(x)$, called the residue field of X at x.

(2) Show that a map of schemes $f\colon X \to Y$ will induce for any point $x \in X$ a natural map of fields $\kappa(f(x)) \to \kappa(x)$.

(3) Show that the underlying set of $Spec(B \otimes_A C)$ is the same as quadruples $(\mathfrak{b}, \mathfrak{c}, \mathfrak{a}, \mathfrak{p})$ where the first three are prime ideals in the corresponding ring with $f^{-1}(\mathfrak{b}) = \mathfrak{a}$ (and similarly for $\mathfrak{c}$) inducing maps of fields $\kappa(\mathfrak{a}) \to \kappa(\mathfrak{b})$ and $\kappa(\mathfrak{a}) \to \kappa(\mathfrak{c})$, and $\mathfrak{p}$ is a prime ideal in $\kappa(\mathfrak{b}) \otimes_{\kappa(\mathfrak{a})} \kappa(\mathfrak{c})$.

(4) Let $f\colon X \to S$ and $g\colon Y \to S$ be maps of schemes, then consider the set of quadruples $\{(x, y, s, \mathfrak{p}) | f(x) = s = g(y), \mathfrak{p} \subset \kappa(x) \otimes_{\kappa(s)} \kappa(y) \text{ prime ideal}\}$. Equip this set with the coarsest topology such that for any pair $(A \to B, A \to C)$ as in (3) and any map of schemes $Spec(B) \subset X$, $Spec(C) \subset Y$, $Spec(A) \subset S$ making all

imaginable diagrams commutative, then the natural map $Spec(B \otimes_A C)$ to the said set of quadruples (via the identification in (3)) is continuous.

(5) Equip the topological space in (4) with a sheaf of rings[2], such that the ringed space is a scheme, which admits maps to both X and Y, with both induced maps to S being the same, and show that these datum makes this scheme the fiber product $X \times_S Y$. In particular, fiber products exist in the category of schemes.

(6) Find an example of $X \to S \leftarrow Y$, such that the topology on $X \times_S Y$ is not the "box topology".

Remark 3.15. Alternatively, one can "glue" $Spec(B \otimes_A C)$'s where these rings come from affine opens in X, S, Y, to produce the fiber product.

Definition 3.16. A map of schemes $f \colon X \to Y$ is called *separated* if the natural map $X \xrightarrow{\Delta=(id,id)} X \times_Y X$ is a closed immersion.

Yet another kind of non-affine schemes.

Exercise 3.17. Design your own example of a scheme which would deserve the name "affine line with double origin". Show that the example you designed is neither quasi-affine nor separated.

(Warning: this is a trick question, so I expect you to modify this question into a "correct" one and solve it.)

Exercise 3.18. (1) Show that a morphism of affine schemes is separated.

(2) Let $X \to Y = Spec(A)$ be a separated morphism from a scheme to an affine scheme. Show that any pair of affine opens $U, V \subset X$ has affine intersection $U \cap V$, moreover the map $\mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.

Definition 3.19 (Properties coming from topology)**.** Let X be a scheme. We say X is connected or quasi-compact or irreducible if its underlying topological space is so.

Recall that a topological space is irreducible if it's nonempty and any pair of nonempty opens inside it has nonempty intersection.

Exercise 3.20. (1) Show that an affine scheme $Spec(A)$ is irreducible iff $\{\text{zerodivisors in } A\} = \{\text{nilpotents in } A\}$.

(2) Show that if a scheme X is irreducible, then the intersection of all opens in X is a singleton, whose element is called the *generic point* of X and usually denoted by $\eta \in X$. Moreover, suppose there is an open affine $Spec(A) \subset X$, describe the local ring $\mathcal{O}_{X,x}$ in terms of A.

Definition 3.21 (Properties coming from commutative algebra)**.** Let P be a property of rings. We say that P is local if the following hold:

- $\forall f \in A$, we have $P(A) \implies P(A[1/f])$;
- $\forall (f_1, \ldots, f_n) = R$, we have $\{\forall i, P(A[1/f_i]\} \implies P(A)$.

Let P be a property of rings. Let X be a scheme. We say X is locally P if $\forall x \in X$, $\exists$ an affine open neighbourhood U of x such that $\mathcal{O}_X(U)$ has property P.

Similarly for properties of A-algebras, and schemes over $Spec(A)$.

Exercise 3.22. Let X be a scheme, let P be a local property of rings. Show that the following are equivalent:

- The scheme X is locally P.
- $\forall U \subset X$ affine, the property $P(\mathcal{O}_X(U))$ holds.
- $\exists$ an affine open cover $X = \cup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.
- $\exists$ an open cover $X = \cup X_j$ such that each X_j is locally P.

Moreover, if X is locally P then every open subscheme is locally P.

Formulate and prove a similar statement for property of A-algebras and schemes over $Spec(A)$.

Exercise 3.23. Show that the properties of being reduced or Noetherian or regular are local. Show that the property of finitely generated as A-algebra is a local property of A-algebras.

Definition 3.24. We say a scheme X is integral if $\forall U \subset X$ affine, the ring $\mathcal{O}_X(U)$ is a domain.

[2]It's not like I'm feeling lazy, but one simply writes again maps of certain form which has appeared repeatedly.

Exercise 3.25. Show that X is integral iff it's both reduced and irreducible. If X is integral, show that the local ring $\mathcal{O}_{X,\eta}$ at the generic point is a field, which is called the *function field of* X.

Definition 3.26. Let k be a field, the category of k-varieties is defined to have objects k-varieties: These are quasi-compact integral schemes together with a map to $Spec(k)$, called the structural map, and moreover the structural map is required to be separated and locally of finite type. Morphisms are defined to be those of schemes compatible with structural maps.

Exercise 3.27. Show that $\mathbb{A}^n_k \coloneqq Spec(k[X_1, \ldots, X_n])$ and $\mathbb{P}^n_k$ are k-varieties.

Exercise 3.28. Let k be a field, and let $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a finite type k-algebra.

(1) Denote the kernel of $A \otimes_k A \xrightarrow{\text{multiplication}} A$ by I, and show that I/I^2 is canonically an A-module

(2)Show that the cokernel of the map $\bigoplus_{i=1}^m A \cdot e_i \xrightarrow{\text{Jacobian matrix}} \bigoplus_{j=1}^n A \cdot dx_j$ is intrinsic by showing that this A-module is canonically identified with I/I^2. This module is denoted by $\Omega^1_{A/k}$.

(3) Let X be a scheme locally finite type over k, show that there is a coherent sheaf on X whose restriction to any affine open $U = Spec(A) \subset X$ is associated with the A-module $\Omega^1_{A/k}$.

Definition 3.29. The coherent sheaf constructed above is called the sheaf of Kähler differentials relative to k, and is denoted by $\Omega^1_{X/k}$ or even short-hand as Ω^1_X when the ground field k is obvious from the context.

Definition 3.30. Let k be a field and let X be a scheme locally finite type over k. We say X/k is smooth if for any $x \in X$, the stalk $\Omega^1_{X/k,x}$ is a free module over $\mathcal{O}_{X,x}$ of rank equal to $\dim(\mathcal{O}_{X,x})$.

Fact 3.31. (1) In the context above, assume furthermore that X is integral. Then being smooth is equivalent to Ω^1_X being locally free of rank equal to the transcendence degree of the function field of X relative to k. (Locally free means, locally on X it's free as an $\mathcal{O}_X$-module.)

(2) Keep notation as above, smooth implies regular. And if k is perfect, regular implies smooth.

4. Cohomology of quasi-coherent sheaves

Exercise 4.1. Find the definition of "complex" and "cohomology".

Construction 4.2. Let A be a ring and let X be a quasi-compact scheme separated over $Spec(A)$. In particular, the intersection of two affine opens in X is again an affine open. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{U} = \{U_i | i \in I\}$ be a finite open covering of X by affine opens. Let $<$ be a total order on the finite indexing set I, define a complex

$$\check{C}(\mathcal{U}, \mathcal{F}) \coloneqq \left(\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i_1 < i_2 \in I^2} \mathcal{F}(U_{i_1} \cap U_{i_2}) \to \ldots \to \prod_{i_1 < i_2 < \ldots < i_r \in I^r} \mathcal{F}(\cap_{\ell=1}^r U_{i_\ell}) \to \ldots \right)$$

Here the differential sends an element (which is itself a collection of elements) $(s_{i_1 < i_2 < \ldots < i_{r-1}})$ in degree $r-1$ to an element in degree r whose spot at $i_1 < i_2 < \ldots < i_r$ is given by the alternating sum $\sum_{j=1}^r (-1)^{j-1} s_{i_1 < i_2 < \ldots < \widehat{i_j} < \ldots < i_r}$. Here $\widehat{(-)}$ means skipping that index, and one should understand the terms of the sum as the restriction of these sections to the smaller open $\cap_{j=1}^r U_{i_j}$.

Exercise 4.3. (1) Show that the above is a complex of A-modules.

(2) Show that if one switches the total order, then the complex one defines would be naturally isomorphic to the one above. (Make sure that you got the signs correct...)

It is in this sense that we view $\check{C}(\mathcal{U}, \mathcal{F})$ as only associated with the open cover and not with the auxiliary total order that we chose from beginning.

Below we shall walk through the reason why the above complex has its cohomology even independent of the affine open cover(!).

Exercise 4.4. (1) Let $X = Spec(A)$ be an affine scheme, let $\mathcal{F}$ be a quasi-coherent sheaf on X, and let $\mathcal{U}$ be a finite open covering by affine opens. Show that the complex $\check{C}(\mathcal{U}, \mathcal{F})$ has no cohomology except in degree 0, which is given by $\mathcal{F}(X)$.

(2) Find the definition of "double complex" and "total cohomology".

(3) Now assume we are in the setup of the above Construction. Suppose we are given two finite open covering by affine opens $\mathcal{U}$ and $\mathcal{V}$. Then form a double complex using $\mathcal{F}(\cap U_i\text{'s} \cap V_j\text{'s})$, such that its total cohomology is isomorphic to both the cohomology of $\check{C}(\mathcal{U}, \mathcal{F})$ and $\check{C}(\mathcal{V}, \mathcal{F})$ via natural maps. Therefore we see that the cohomology A-modules we get are independent of the open covering by affine opens.

(Hint: use (1).)

Definition 4.5. In the setup of the above Construction, we define $H^i(X, \mathcal{F}) \coloneqq H^i(\check{C}(\mathcal{U}, \mathcal{F}))$ for some (hence any, by the above) finite open covering by affine opens $\mathcal{U}$.

Exercise 4.6. (1) Search online for the following expressions:"long exact sequence" and "snake lemma".

(2) Given an exact sequence of quasi-coherent sheaves on a scheme X

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{C} \to 0.$$

Show that we get a natural induced long exact sequence of cohomologies:

$$\cdots \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G}) \to H^i(X, \mathcal{C}) \to H^{i+1}(X, \mathcal{F}) \to \cdots$$

This concludes the first part of our course.

5. Things that I wished to cover if I had time

I thought I would write some subsets of items in Hartshorne's "Algebraic Geometry" Chapter II & III, but no, after looking back at it, I think that does contain minimum requirement for AG.

Email address: `lishizhang@amss.ac.cn`